

1/1

STUDY MATERIAL (with assignments)
for

B.Sc. 2nd Semester, Statistics (Honours)

Paper : STA-HG-2016 . Unit : 2 (ii)

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Joint, marginal and conditional distributions : -
(Continuous case)

Joint probability density function : A function $f(x, y)$

of two continuous random variables X and Y taking values in the intervals $[a, b]$ and $[c, d]$ respectively is called the joint probability density function if

i) $f(x, y) \geq 0 \quad \forall x \in [a, b], y \in [c, d]$

ii) $\iint_{a c}^{b d} f(x, y) dx dy = 1$

and iii) $\iint f(x, y) dx dy = P_r \left[x - \frac{1}{2} dx \leq X \leq x + \frac{1}{2} dx, y - \frac{1}{2} dy \leq Y \leq y + \frac{1}{2} dy \right]$

In general, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

Joint probability distribution function :

For two continuous r.v. X and Y taking values in $(-\infty, \infty)$ the joint probability distribution function is denoted by $F(x, y)$ is defined as

$$F(x, y) = P(-\infty < X \leq x, -\infty < Y \leq y)$$
$$= \int_{-\infty}^x \int_{-\infty}^y f(x, y) dx dy$$

where $f(x, y)$ is the joint p.d.f. of (X, Y) .

Marginal probability density functions or marginal distributions:

Given the joint pdf $f(x,y)$ of r.v. X and Y the marginal probability density function of X is defined as

$$g(x) = \int_{-\infty}^{\infty} f(x,y) dy, \quad -\infty < x < \infty$$

$$\text{where } \int_{-\infty}^{\infty} g(x) dx = \iint_{-\infty}^{\infty} f(x,y) dx dy = 1$$

Similarly the marginal probability density function of r.v. Y is defined as

$$h(y) = \int_{-\infty}^{\infty} f(x,y) dx, \quad -\infty < y < \infty$$

$$\text{so that } \int_{-\infty}^{\infty} h(y) dy = \iint_{-\infty}^{\infty} f(x,y) dx dy = 1$$

Marginal distribution functions:

The marginal distribution function of r.v. X given the joint pdf $f(x,y)$ is given by

$$F(x) = \int_{-\infty}^x \left[\int_{-\infty}^{\infty} f(x,y) dy \right] dx$$

and the marginal distribution function of r.v. Y is defined as

$$F(y) = \int_{-\infty}^y \left[\int_{-\infty}^{\infty} f(x,y) dx \right] dy$$

Again if the joint probability distribution function $F(x,y)$ is given then

$$F(x) = \frac{d}{dy} F(x,y)$$

$$\text{and } F(y) = \frac{d}{dx} F(x,y).$$

Conditional density functions & or conditional prob. distributions:

Given the joint pdf $f(x, y)$ of the continuous r.v. X and Y the conditional density function of X for given $Y=y$ is defined as

$$f(x/y) = \frac{f(x, y)}{h(y)}, \quad h(y) > 0$$

where $h(y) = \int_{-\infty}^{\infty} f(x, y) dx$ is the marginal density of Y .

8 Clearly we have

$$\int_{-\infty}^{\infty} f(x/y) dx = \int_{-\infty}^{\infty} \frac{f(x, y)}{h(y)} dx = \frac{h(y)}{h(y)} = 1$$

Similarly the conditional density of Y given $X=x$ is defined as

$$f(y/x) = \frac{f(x, y)}{g(x)}, \quad g(x) > 0$$

where $g(x) = \int_{-\infty}^{\infty} f(x, y) dy$ is the marginal density of X .

Here also we have

$$\int_{-\infty}^{\infty} f(y/x) dy = \int_{-\infty}^{\infty} \frac{f(x, y)}{g(x)} dy = \frac{g(x)}{g(x)} = 1$$

Independence of random variables :

Two continuous r.v. X and Y will be called independent if

$$f(x, y) = g(x) h(y)$$

\Rightarrow joint probability density function

= Product of marginal density functions

Problem: The joint pdf of (X, Y) is as given below:

$$f(x, y) = k(4-2x+y), \quad 0 < x < 3, \quad 2 < y < 4 \\ = 0 \quad \text{elsewhere}$$

- Find (i) k (ii) The marginal distributions of X and Y
 (iii) The conditional distribution of X given $Y=y$
 (iv) " " " " " " " Y " $X=x$
 (v) $P(X < 2 \cap Y < 3)$, (vi) $P(X+Y < 4)$
 (vii) $P(X < 2 | Y < 3)$

Solution:

Since

$$(i) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

$$\Rightarrow \int_0^3 \int_2^4 k(4-2x+y) dx dy = 1$$

$$\text{or, } k \int_0^3 \left[\int_2^4 (4-2x+y) dy \right] dx = 1$$

$$\text{or, } k \int_0^3 \left[(4-2x)y + \frac{y^2}{2} \right]_2^4 dx = 1$$

$$\text{or, } k \int_0^3 \left[(4-2x)(4-2) + \frac{4^2 - 2^2}{2} \right] dx = 1$$

$$\text{or, } k \int_0^3 (8-4x+6) dx = 1$$

$$\text{or, } k \left[14x - 2x^2 \right]_0^3 = 1$$

$$\Rightarrow k = \frac{1}{24}$$

$$\therefore f(x, y) = \frac{1}{24}(4-2x+y), \quad 0 < x < 3, \quad 2 < y < 4 \\ = 0 \quad \text{elsewhere.}$$

(ii) The marginal distribution of X is

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_2^4 \frac{1}{24} (4-2x+y) dy$$

(contd.)

$$\begin{aligned}g(x) &= \frac{1}{24} \left[4y - 2xy + \frac{y^2}{2} \right]_2^4 \\&= \frac{1}{24} (16 - 8x + 8 - 8 + 4x - 2) \\&= \frac{1}{24} (14 - 4x), \quad 0 < x < 3\end{aligned}$$

Again the marginal pdf of Y is

$$\begin{aligned}h(y) &= \int_{-\infty}^{\infty} f(x,y) dx = \int_0^3 \frac{1}{24} (4 - 2x + y) dx \\&= \frac{1}{24} \left[4x - 2\frac{x^2}{2} + yx \right]_0^3 \\&= \frac{1}{24} (12 - 9 + 3y) \\&= \frac{1}{8} (1 + y), \quad 2 < y < 4\end{aligned}$$

(iii) The conditional density of X given $Y=y$ is

$$\begin{aligned}f(x|y) &= \frac{f(x,y)}{h(y)} \\&= \frac{\frac{1}{24} (4 - 2x + y)}{\frac{1}{8} (1 + y)} \\&= \frac{1}{3} \frac{(4 - 2x + y)}{1 + y}, \quad 0 < x < 3\end{aligned}$$

(iv) The conditional density of Y given $X=x$ is

$$\begin{aligned}f(y|x) &= \frac{f(x,y)}{g(x)} \\&= \frac{\frac{1}{24} (4 - 2x + y)}{\frac{1}{24} (14 - 4x)} \\&= \frac{1}{2} \frac{(4 - 2x + y)}{14 - 2x}, \quad 2 < y < 4\end{aligned}$$

$$\begin{aligned}(v) P(X < 2 \cap Y > 3) &= \int_0^2 \int_2^3 f(x,y) dx dy \\&= \frac{1}{24} \int_0^2 \left[\int_2^3 (4 - 2x + y) dy \right] dx\end{aligned}$$

(Contd.)

$$\begin{aligned}
 &= \frac{1}{24} \int_0^2 \left[(4-2x) y + \frac{y^2}{2} \right]_2^3 dx \\
 &= \frac{1}{24} \int_0^2 \left(4 - 2x + \frac{5}{2} \right) dx \\
 &= \frac{1}{24} \left[\frac{13}{2}x - x^2 \right]_0^2 = \frac{1}{24}(9) = \frac{3}{8}
 \end{aligned}$$

$$(vii) P(X+Y < 4) = \frac{1}{24} \int_0^2 \left[\int_2^{4-x} (4-2x+y) dy \right] dx$$

$\because x < 4-2$ and min^m value of $Y=2$, so max value of x is 2]

$$\begin{aligned}
 &= \frac{1}{24} \int_0^2 \left[(4-2x)y + \frac{y^2}{2} \right]_2^{4-x} dx \\
 &= \frac{1}{24} \int_0^2 \left[(4-2x)(2-x) + \frac{1}{2}((4-x)^2 - 4) \right] dx \\
 &= \frac{1}{48} \int_0^2 (5x^2 - 24x + 28) dx = \frac{4}{9} \quad (\text{After simplification})
 \end{aligned}$$

$$(viii) P[X < 2 | Y < 3] = \frac{P(X < 2 \cap Y < 3)}{P(Y < 3)}$$

$$\text{Now } P(X < 2 \cap Y < 3) = \frac{3}{8} \quad [\text{From (v)}]$$

$$\begin{aligned}
 \text{and } P(Y < 3) &= \int_0^2 h(y) dy \\
 &= \frac{1}{8} \int_0^3 (1+y) dy \quad \text{using results of (i)} \\
 &= \frac{1}{8} \left[y + \frac{y^2}{2} \right]_0^3 \\
 &= \frac{7}{16}
 \end{aligned}$$

$$\therefore P(X < 2 | Y < 3) = \frac{\frac{3}{8}}{\frac{7}{16}} = \frac{6}{7}$$

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Home assignments (06/05/2020)

Prepare answers to the following questions:

1. Define joint, marginal and conditional distributions of discrete random variables.
2. Explain joint density function of two continuous random variables. Also define marginal and conditional densities. When are two random variables said to be independent?
3. Two discrete r.v. X and Y has the bivariate distribution as follows:

$$p(x,y) = \frac{1}{27}(2x+y) \text{ where } X \text{ and } Y \text{ can assume only the integer values 0, 1 and 2.}$$

Find the conditional distributions of Y given X=x.

4. The joint p.d.f. of X and Y is given by

$$f(x,y) = \begin{cases} c & ; 0 < x < 1, 0 < y < x \\ 0 & , \text{elsewhere.} \end{cases}$$

- i) Find the value of c.
 - ii) Find the marginal distributions of X and Y.
 - iii) Find the conditional distⁿ of X given Y=y.
 - iv) Are X and Y independent?
5. The joint pdf of X and Y is given by

$$f(x,y) = 4xy e^{-(x+y)} ; x \geq 0, y \geq 0$$

Examine whether X and Y are independent.

Submit the answers offline.